

# Universal description of viscoelasticity with foliation preserving diffeomorphisms\*

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## Abstract

We review our recent proposal for a universal description of generic single-component viscoelastic systems with a single relaxation time. Foliation preserving diffeomorphisms are introduced as an underlying symmetry which naturally interpolates between the two extreme characters of elasticity and fluidity. The symmetry is found to be powerful enough to determine the dynamics in the first order of strains.

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# 1 Introduction

Viscoelasticity (see, e.g., [1, 2]) is a unifying notion which contains solid and fluid as limiting cases. It was defined by James Clerk Maxwell such as to describe the materials that behave as elastic solids at short times scales and as viscous fluids at long time scales [3, 4].

In order to grasp the idea of this definition, we consider a material (like a chewing gum) consisting of many particles bonding each other. They stay at their equilibrium positions in the absence of stresses (as in the leftmost of Fig. 1), and we now suppose that an external force is applied to deform the material. An internal stress is then produced in the body, and according to the definition, this internal stress can be treated as an elastic force at least during short intervals of time. However, if we keep the deformation much longer than some period of time (characteristic to each material and to be called the relaxation time), then the bonding structure changes to reduce the free energy, and the internal stress vanishes eventually. The point here is that the two configurations (the center and the rightmost of Fig. 1) exhibit the same shape (i.e. same configurations for the positions of particles) but have different bonding structures.

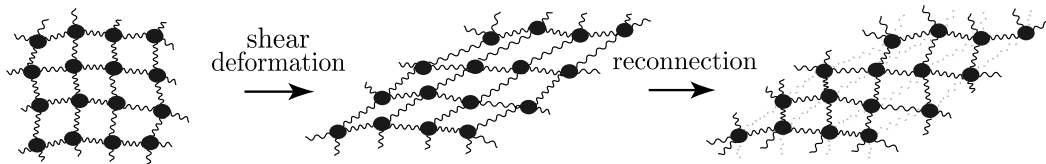


Figure 1: Shear deformation of a viscoelastic material

We thus see that the bonding structure itself is dynamical for viscoelastic materials, and is lead to introduce a new dynamical variables (to be called *intrinsic metric* [5]) to specify the bonding structure. It was claimed in [5] that viscoelasticity may have a universal description if we define the intrinsic metric properly and assume the invariance of the system under *foliation preserving diffeomorphisms* (FPDs). In fact, it was shown there that FPDs naturally interpolate between the two extreme limits of elasticity and fluidity, and are powerful enough to determine the dynamics in the first order of strains.

The main purpose of the present article is to give a brief review of [5] in a concise way, and also to present some additional stuffs which were not mentioned in [5].

Our discussions will be constrained to single-component materials. Note that, even for this case, there can be two relaxation times, one being for shear deformations and another for bulk compressions (or expansions). In this article, we set the relaxation time for bulk

compressions to be infinite, so that the bulk stress does not undergo relaxations (see Fig. 2). This ensures that the system exhibits fluidity at long intervals of time.

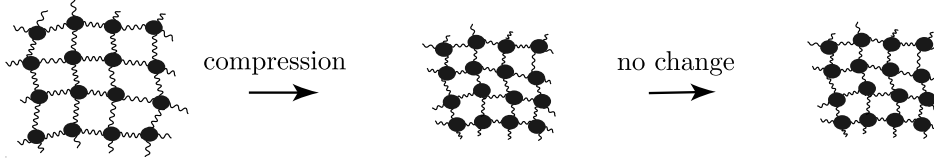


Figure 2: Compression of a viscoelastic material

## 2 Geometrical setup

We describe a viscoelastic material within the framework of world-volume theory with the world-volume coordinates  $\xi = (\xi^a)$  ( $a = 1, 2, 3$ ) (labeling the material particles of a body) and with the target space coordinates  $X = (X^i)$  ( $i = 1, 2, 3$ ). The shape of the material at time  $t$  is then specified by giving the functions  $X^i(\xi, t)$ . In order to further describe the bonding structure, we introduce an intrinsic metric  $\bar{h}_{ab}(\xi, t)$  as follows.

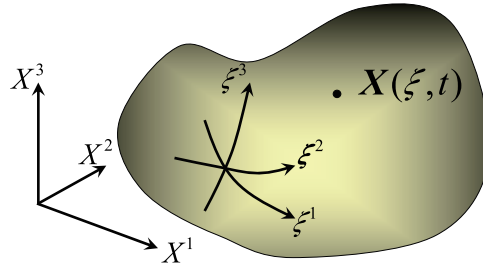


Figure 3: World-volume of a viscoelastic material

We first recall that our viscoelastic materials are defined to be elastic at least for a very short time. This means that for any (sufficiently small) portion in a material, its *natural shape* can be defined at each moment as the shape that would be taken when all stresses were virtually removed (see Fig. 4).

This observation enables us to define two kinds of metrics on the world-volume at each time  $t$  (see Fig. 5). The first one is the *induced metric*  $h_{ab}(\xi, t)$  which measures the length in the real shape:

$$ds^2 = h_{ab}(\xi, t) d\xi^a d\xi^b, \quad (1)$$

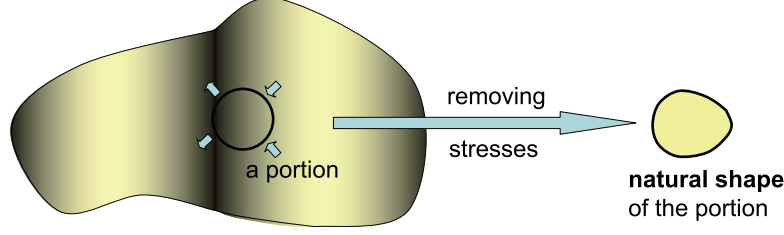


Figure 4: Definition of the natural shape for a small portion

where

$$h_{ab}(\xi, t) \equiv e_a^i(\xi, t) e_b^i(\xi, t) \quad (e_a^i(\xi, t) \equiv \partial_a X^i(\xi, t) : \text{dreibein}). \quad (2)$$

Another is the *intrinsic metric*  $\bar{h}_{ab}(\xi, t)$  which measures the length in the natural shape:

$$d\bar{s}^2 = \bar{h}_{ab}(\xi, t) d\xi^a d\xi^b. \quad (3)$$

The difference between them defines the *strain tensor*,

$$\varepsilon_{ab}(\xi, t) \equiv \frac{1}{2} (h_{ab}(\xi, t) - \bar{h}_{ab}(\xi, t)). \quad (4)$$

In fact, for an elastic material at rest without stress, we can take the coordinates  $\xi = (\xi^a)$  such that  $\bar{h}_{ab} = \delta_{ab}$ . Then, when the material is slightly deformed under stresses,  $X^a(\xi, t) = \xi^a + u^a(\xi, t)$  ( $u^a$ : displacements), we have  $h_{ab} = \partial_a X^i \partial_b X^i = \delta_{ab} + \partial_a u_b + \partial_b u_a + O(u^2)$ , so that  $\varepsilon_{ab} = (1/2)(\partial_a u_b + \partial_b u_a) + O(u^2)$ .

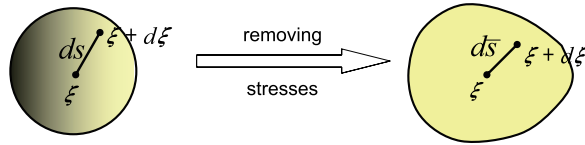


Figure 5: The induced and intrinsic metrics

We denote by  $\rho_0$  the mass density in the absence of strains. In this article, we assume that  $\rho_0$  is a constant independent of  $\xi$  and  $t$ . Then the mass contained in a volume element  $d^3\xi = d\xi^1 d\xi^2 d\xi^3$  is given by  $\rho_0 \sqrt{h} d^3\xi$ . The mass density  $\rho$  in the real three-dimensional space is given by the mass conservation  $\rho \sqrt{h} d^3\xi = \rho_0 \sqrt{h} d^3\xi$  as  $\rho = (\sqrt{h}/\sqrt{h}) \rho_0$ .

Let us consider various deformations of a 2D material. Here  $(\xi^1, \xi^2)$  are comoving coordinates attached to material particles.

**(a) shear deformations:** Suppose that a material in equilibrium is deformed with an external force as in Fig. 6. The two metrics  $h_{ab}$  and  $\bar{h}_{ab}$  agree before the deformation. When the material is deformed, the induced metric  $h_{ab}$  varies, expressing the change of the shape. In contrast, the intrinsic metric  $\bar{h}_{ab}$  retains its original form just after the deformation. This is because the material should exhibit elasticity for a short period of time, and the natural shape at that moment is its original form in equilibrium. At sufficiently later times,  $\bar{h}_{ab}$  gradually changes its values and comes to equal the induced metric.

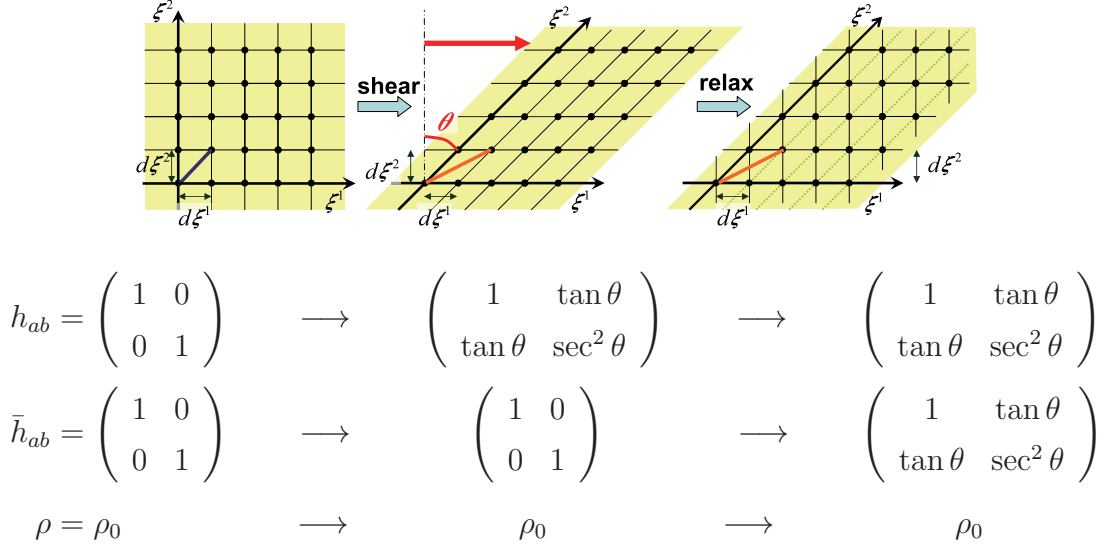


Figure 6: Shear deformation with the angle  $\theta$

**(b) bulk compressions:** Suppose that a material in equilibrium is compressed with a factor  $\lambda (< 1)$  as in Fig. 7. The two metrics  $h_{ab}$  and  $\bar{h}_{ab}$  again agree before the deformation, and the induced metric  $h_{ab}$  changes after the deformation. However, according to our assumption that the bulk compression (or expansion) does not undergo relaxation, the intrinsic metric  $\bar{h}_{ab}$  retains its original form.

### 3 Foliation preserving diffeomorphisms

So far the world-volume coordinates  $\xi$  have been attached to a material, labeling the material particles. However, we can introduce arbitrary coordinates  $\xi$  at each time  $t$ , independently of the labeling of material particles. In this case we need to specify the relative motion of the material to the coordinates, but this can be carried out by introducing a vector field  $\mathcal{N} = N^a(\xi, t)\partial_a$  (see Fig. 8) [5]. More precisely, the vector field is defined

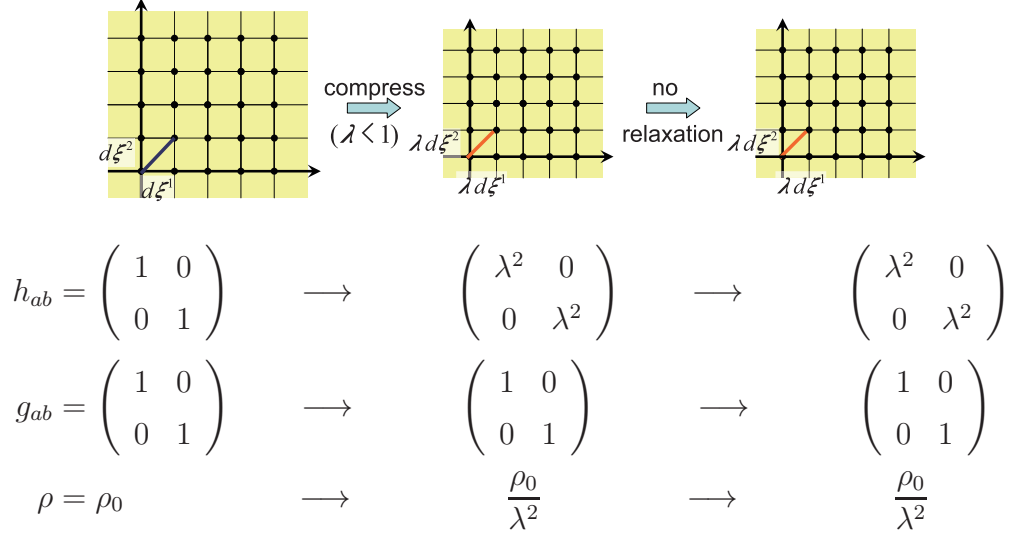


Figure 7: Compression by the factor  $\lambda$  ( $\lambda < 1$ )

such that the material particle located at  $\xi$  at time  $t$  is supposed to move to the position  $\xi^a + N^a(\xi, t)dt$  after the time interval  $dt$ .

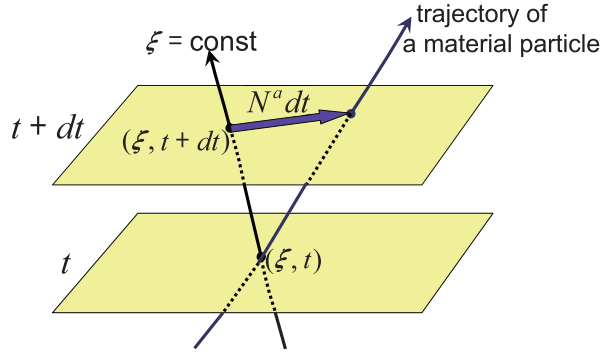


Figure 8: Definition of  $N^a$

With  $N^a(\xi, t)$ , the system comes to have the gauge symmetry of *foliation preserving diffeomorphisms* (FPDs), reflecting the redundancy in the description. The set of FPDs consist of the  $(3 + 1)$ -dimensional reparametrizations that do not move the time slices, and is generated by the infinitesimal transformations of the form (see [5] for references of mathematical details):

$$\delta\xi^a(\xi, t) = \epsilon^a(\xi, t), \quad \delta t(\xi, t) = 0. \quad (1)$$

The fields  $X^i$ ,  $h_{ab}$  and  $\bar{h}_{ab}$  are intrinsically three-dimensional and thus transform covariantly

as

$$\delta X^i = \epsilon^a \partial_a X^i, \quad (2)$$

$$\delta h_{ab} = \partial_a \epsilon^c h_{cb} + \partial_b \epsilon^c h_{ac} + \epsilon^c \partial_c h_{ab}, \quad (3)$$

$$\delta \bar{h}_{ab} = \partial_a \epsilon^c \bar{h}_{cb} + \partial_b \epsilon^c \bar{h}_{ac} + \epsilon^c \partial_c \bar{h}_{ab}, \quad (4)$$

while the transformation of the shift vector has an additional inhomogeneous term as

$$\delta N^a = -\partial_c \epsilon^a N^c + \epsilon^c \partial_c N^a - \dot{\epsilon}^a. \quad (5)$$

The time derivatives of a FPD-covariant tensor  $T_{a_1 a_2 \dots}^{b_1 b_2 \dots}$  are no longer covariant, but can be made so by introducing the covariant derivative [5]:

$$\frac{D}{Dt} T_{a_1 a_2 \dots}^{b_1 b_2 \dots} \equiv \frac{\partial}{\partial t} T_{a_1 a_2 \dots}^{b_1 b_2 \dots} + \mathcal{L}_{\mathcal{N}} T_{a_1 a_2 \dots}^{b_1 b_2 \dots}, \quad (6)$$

where  $\mathcal{L}_{\mathcal{N}}$  is the Lie derivative with respect to  $\mathcal{N} = N^a \partial_a$ .

For example, the velocity  $v^i(\xi, t)$  and the acceleration  $a^i(\xi, t)$  of the material particle located at  $(\xi, t)$  are given, respectively, by

$$v^i = \frac{DX^i}{Dt} = \dot{X}^i + N^a \partial_a X^i, \quad a^i = \frac{Dv^i}{Dt} = \frac{D^2 X^i}{Dt^2} = \dot{v}^i + N^a \partial_a v^i. \quad (7)$$

This can be easily confirmed by noting that the material particle at  $(\xi, t)$  moves after the time interval  $dt$  to the point  $(\xi + N dt, t + dt)$ :

$$X^i(\xi + N dt, t + dt) = X^i(\xi, t) + [\dot{X}^i(\xi, t) + N^a(\xi, t) \partial_a X^i(\xi, t)] dt \equiv X^i(\xi, t) + v^i(\xi, t) dt. \quad (8)$$

As for the metrics  $h_{ab}$  and  $\bar{h}_{ab}$ , their covariant derivatives are given, respectively, by

$$K_{ab} \equiv \frac{1}{2} \frac{Dh_{ab}}{Dt} = \frac{1}{2} (\dot{h}_{ab} + \partial_a N^c h_{cb} + \partial_b N^c h_{ac} + N^c \partial_c h_{ab}), \quad (9)$$

$$\bar{K}_{ab} \equiv \frac{1}{2} \frac{D\bar{h}_{ab}}{Dt} = \frac{1}{2} (\dot{\bar{h}}_{ab} + \partial_a N^c \bar{h}_{cb} + \partial_b N^c \bar{h}_{ac} + N^c \partial_c \bar{h}_{ab}), \quad (10)$$

and are called the extrinsic curvatures. Note that

$$K_{ab} - \bar{K}_{ab} = \frac{D\varepsilon_{ab}}{Dt}, \quad (11)$$

and

$$K_{ab} = \frac{1}{2} (\nabla_a v_b + \nabla_b v_a), \quad (12)$$

where  $v_a \equiv e_a^i v^i$  and  $\nabla_a$  is the covariant derivative with respect to  $h_{ab}$  [5].

As was stated above, the set of FPDs forms a gauge symmetry group of the system, and expresses a redundancy in the description of the system. This can be gauge-fixed arbitrarily according to convenience in describing the dynamics of a given system. Two of the useful gauge fixings are the followings:

[**Comoving frame**] This is useful in describing elastic materials and is defined by the condition  $N^a(\xi, t) = 0$ , which leads to  $v^i(\xi, t) = \dot{X}^i(\xi, t)$ .

[**Laboratory frame**] This is useful in describing fluids and is defined by the condition  $X^a(\xi, t) \equiv \xi^a$ , from which follow the equations  $v^a = N^a$ ,  $h_{ab} = \delta_{ab}$ ,  $\rho = \rho_0 \sqrt{\bar{h}}$ , and  $a^a = \dot{v}^a + v^b \partial_b v^a$ . The last equation shows that the covariant derivative in this frame becomes the material (Lagrangian) derivative.

## 4 Fundamental equations

We now write down a set of equations which determine the time evolution of twelve variables  $X^i$  (three),  $\bar{h}_{ab}$  (six) and  $N^a$  (three) up to FPDs (three). Our strategy is to first write down equations which hold in both of the elastic and fluid limits (this can be done in specific frames), and then to make the equations covariant under FPDs.

### 4.1 Equation of motion for $\bar{h}_{ab}(\xi, t)$

We first consider the limiting case of elasticity in the comoving frame ( $N^a = 0$ ). We then have  $\dot{\bar{h}}_{ab} = 0$  because the natural shape does not evolve in time for elastic materials. Its FPD-covariant expression is  $\bar{K}_{ab} = 0$ , and thus we expect that nonvanishing  $\bar{K}_{ab}$  represents genuinely plastic (i.e. nonelastic) deformations. However, one can show that its trace part  $\bar{K} \equiv \bar{h}^{ab} \bar{K}_{ab}$  always vanishes due to the mass conservation [5]:

$$\bar{K} = 0. \quad (1)$$

In fact, in the laboratory frame,  $\bar{K}$  is expressed as  $(1/\rho)(\dot{\rho} + \partial_a(\rho v^a))$  [5] and thus must vanish.

The above consideration shows that only the traceless (shear) component of the extrinsic curvature  $\bar{K}_{ab}$  expresses the changing rate of the bonding structure, which should be proportional to shear strains in the following form [5]:

$$\bar{K}_{ab} - \frac{1}{3} \bar{K} \bar{h}_{ab} = \frac{1}{\tau} \left( \varepsilon_{ab} - \frac{1}{3} (\bar{h}^{cd} \varepsilon_{cd}) \bar{h}_{ab} \right). \quad (2)$$



Note that the second term on the left-hand side can actually be omitted because  $\bar{K} = 0$ . The inverse of the proportional coefficient,  $\tau$ , has the dimension of time, and should correspond to the relaxation time.

In order to check the consistency of this equation (named the *rheology equation* in [5]), we take two extreme limits. In the elastic limit ( $\tau \rightarrow \infty$ ), we obtain  $\bar{K}_{ab} = 0$ , which again means that there occurs no time evolution of intrinsic metric. In contrast, in the fluid limit ( $\tau \rightarrow 0$ ), we have  $\varepsilon_{ab} \propto \bar{h}_{ab}$ , or  $h_{ab} \propto \bar{h}_{ab}$ . This means that the material acquires only homogeneous compression around each point, which is actually the characteristic property defining fluid.

## 4.2 Equation of motion for $X^i(\xi, t)$

The dynamics of  $X^i(\xi, t)$  should be expressed as Euler's equation

$$\rho a_a = -\nabla^b T_{ba}. \quad (3)$$

The leading form of the stress tensor  $T_{ab}$  in the derivative expansion can be determined by the following requirements:

- $T_{ab}$  be symmetric and covariant under FPDs.
- $T_{ab}$  be linear in the strain  $\varepsilon_{ab}$  and the spatial derivative of the velocity,  $\nabla_a v_b$ .

These requirements imply that  $T_{ab}$  is a linear combination of the irreducible components of  $\varepsilon_{ab}$  and  $K_{ab} = (1/2)(\nabla_a v_b + \nabla_b v_a)$  (see (12)):

$$T_{ab} = -2\mu \varepsilon'_{ab} - \frac{1}{\kappa} \varepsilon h_{ab} - 2\gamma K'_{ab} - \zeta K h_{ab}, \quad (4)$$

where

$$\varepsilon \equiv h^{ab} \varepsilon_{ab}, \quad \varepsilon'_{ab} \equiv \varepsilon_{ab} - \frac{1}{3} \varepsilon h_{ab}, \quad K \equiv h^{ab} K_{ab}, \quad K'_{ab} \equiv K_{ab} - \frac{1}{3} K h_{ab}. \quad (5)$$

The set of equations (1), (2), (3) and (4) give nine independent equations, and thus can describe the time evolution of twelve dynamical variables up to FPDs. In order to see the meaning of the coefficients in (4), we consider two extreme limits of elasticity and fluidity. In the elastic limit, we have  $\bar{K}_{ab} = 0$  and thus  $K'_{ab} = (D\varepsilon_{ab}/Dt)(1 + O(\varepsilon))$ ,  $K = (D\varepsilon/Dt)(1 + O(\varepsilon))$ . Then the stress tensor becomes

$$T_{ab} = -2\mu \varepsilon'_{ab} - \frac{1}{\kappa} \varepsilon h_{ab} - 2\gamma \frac{D\varepsilon'_{ab}}{Dt} - \zeta \frac{D\varepsilon}{Dt} h_{ab} + O(\varepsilon^2), \quad (6)$$

which shows that the four terms represent, respectively, shear and bulk moduli, and shear and bulk frictions. On the other hand, the fluid limit is realized by considering at time scales much longer than the relaxation time  $\tau$ . By using the rheology equation in the form  $\varepsilon'_{ab} = \tau K'_{ab}(1 + O(\varepsilon))$ , Eq. (4) becomes

$$T_{ab} = -2\eta K'_{ab} - \zeta K h_{ab} - \frac{1}{\kappa} \varepsilon h_{ab} + O(\varepsilon^2), \quad (7)$$

where  $\eta = \gamma + \mu\tau$  and  $\zeta$  are shear and bulk viscosities. The last term expresses the pressure with  $\kappa$  being the bulk compressibility.

## 5 Dispersion relations

In order to get dispersion relations for the set of equations (1)–(4), we expand solutions around the hydrostatic solution in the laboratory frame ( $X^a(\xi, t) = \xi^a$ ,  $h_{ab} = \delta_{ab}$ ) as

$$\bar{h}_{ab} = \delta_{ab} + \Delta \bar{h}_{ab}, \quad N^a = 0 + \Delta N^a. \quad (1)$$

We consider plane waves propagating in the  $X^3(= \xi^3)$ -direction:

$$\Delta \bar{h}_{ab}(\xi, t) = \Delta \bar{h}_{ab}(k, \omega) e^{ik\xi^3 - i\omega t}, \quad \Delta N^a(\xi, t) = \Delta N^a(k, \omega) e^{ik\xi^3 - i\omega t}. \quad (2)$$

The equations can be separated according to the helicity, and we have the following equations:

- helicity 0 (shear modes)  $\{\Delta N^1 \pm i\Delta N^2, \Delta \bar{h}_{31} \mp i\Delta \bar{h}_{32}\}$ :

$$\omega^3 + i\left[\frac{1}{\tau} + \frac{1}{\rho_0}\left(\zeta + \frac{4}{3}\gamma\right)k^2\right]\omega^2 - \frac{1}{\rho_0}\left[\frac{1}{\tau}\left(\zeta + \frac{4}{3}\gamma\right) + \left(\frac{1}{\kappa} + \frac{4}{3}\mu\right)\right]k^2\omega - \frac{i}{\rho_0\kappa\tau}k^2 = 0 \quad (3)$$

$$\text{fluid limit } (\tau \rightarrow 0): \quad \omega^2 + \frac{i}{\rho_0}\left(\zeta + \frac{4}{3}\gamma\right)k^2\omega - \frac{1}{\rho_0\kappa}k^2 = 0 \quad (4)$$

$$\text{elastic limit } (\tau \rightarrow \infty): \quad \omega^2 + \frac{i}{\rho_0}\left(\zeta + \frac{4}{3}\gamma\right)k^2\omega - \frac{1}{\rho_0}\left(\frac{1}{\kappa} + \frac{4}{3}\mu\right)k^2 = 0 \quad (5)$$

- helicity 1 (sound modes)  $\{\Delta N^3, \Delta \bar{h}_{33}, \Delta \bar{h}_{11} + \Delta \bar{h}_{22}\}$ :

$$\omega^2 + i\left(\frac{1}{\tau} + \frac{\gamma}{\rho_0}k^2\right)\omega - \frac{1}{\rho_0}\frac{\gamma + \mu\tau}{\tau}k^2 = 0 \quad (6)$$

$$\text{fluid limit } (\tau \rightarrow 0): \quad i\omega - \frac{\eta}{\rho_0}k^2 = 0 \quad (7)$$

$$\text{elastic limit } (\tau \rightarrow \infty): \quad \omega^2 + i\frac{\gamma}{\rho_0}k^2\omega - \frac{\mu}{\rho_0}k^2 = 0 \quad (8)$$

- helicity 2 (scalar modes)  $\{\Delta \bar{h}_{11} - \Delta \bar{h}_{22} \mp 2i\Delta \bar{h}_{12}\}$ :

$$\omega = -\frac{i}{\tau} \quad (9)$$

## 6 Conclusion and discussion

In this work we have shown that viscoelasticity can have a universal description by introducing an intrinsic metric to express bonding structures and by adopting FPDs as an underlying gauge symmetry. We have seen that FPDs naturally interpolate between elasticity and fluidity, and that covariance under FPDs uniquely determines the dynamics in the first order of strains.

The next step should be to extend our framework to the systems with more than one relaxation time and with multiple components, in order to describe more realistic materials.

We conclude with a comment that in the sound modes the diffusion equation becomes a telegraph-like equation when the relaxation time is taken finite (see Eq. (6)). This means that information propagates only inside the “light cone”, and thus the causality problem (inherent in diffusion equations) disappears when the effect of elasticity is taken into account. This property should be taken over also in the “relativistic extension of viscoelasticity” [6], and would serve as a causal completion of relativistic fluid mechanics.

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